STAT 821 HOMEWORK 2 SOLUTION

Question 5.10

(a) x: number of failures before getting the mth success.

$$f(x) = \binom{m+x-1}{m-1} p^m (1-p)^x I_{(0,1,2,\dots)} \qquad (*)$$

There are m-1 success out of the first m+x-1 trials, the probability of which is $\binom{m+x-1}{m-1} p^m (1-p)^x$. And the last trial must be a success, with probability p. And trials are independent to each other, so result in (*)

(b)

$$\sum_{x=0}^{\infty} \binom{m+x-1}{m-1} p^m (1-p)^x = p^m \sum_{x=0}^{\infty} \binom{m+x-1}{m-1} (1-p)^x$$

We know

$$\begin{aligned}
\binom{-m}{x} &= \frac{(-m)(-m-1)\dots(-m-k)\dots}{x!(-m-x)(-m-x-1)\dots(-m-x-k)\dots} \\
&= \frac{m(m+1)\dots(m+x-1)(-1)^x}{x!} \\
&= \binom{m+x-1}{m-1} (-1)^x
\end{aligned}$$

So

$$\begin{aligned}
& p^m \sum_{x=0}^{\infty} \binom{m+x-1}{m-1} (1-p)^x \\
= \; & p^m \sum_{x=0}^{\infty} \binom{-m}{x} (-1)^x (1-p)^x \\
= \; & p^m (1+p-1)^{-m} \\
= \; & 1
\end{aligned}$$

(c)

$$f(x) = \binom{m+x-1}{m-1} e^{m \log p + x \log(1-p)}$$

is in form of (5.1) with

$$\eta(p) = \log(1-p) \quad B(p) = -m\log(p) \quad h(x) = \binom{m+x-1}{m-1}$$

Thus it is one-parameter exponential family.

(d) Write the pdf in canonical form

$$f(x) = \binom{m+x-1}{m-1} e^{x\log(1-p)+m\log(1-e^{\eta})}$$

where

$$\eta = \log(1-p) \quad T = X \quad A = -m\log(1-e^{\eta})$$

By theorem 5.10,

$$M_x(u) = \frac{e^{A(\eta+u)}}{e^{A(\eta)}} = \left[\frac{1-e^{\eta}}{1-e^{\eta+u}}\right]^m$$

so

$$M_x(u) = \left[\frac{p}{1-(1-p)e^u}\right]^m$$

(e) An application of theorem 5.8 gives

$$\begin{aligned} E(X) &= \frac{d}{d\eta}A(\eta) \\ &= -m\frac{-e^{\eta}}{1-e^{\eta}} \\ &= \frac{m(1-p)}{p} \end{aligned}$$

$$\begin{aligned} Var(X) &= \frac{d^2}{d\eta^2}A(\eta) \\ &= \frac{me^{\eta}}{(1-e^{\eta})^2} \\ &= \frac{m(1-p)}{p^2} \end{aligned}$$

Question 5.12

$$P(X = x) = \frac{a(x)\theta^x}{c(\theta)} = a(x)e^{x\log\theta - \log c(\theta)}$$

Let

$$\eta = \log\theta \quad \theta = e^{\eta} \quad c(\theta) = c(e^{\eta}) \quad T = x \quad A(\eta) = \log c(e^{\eta})$$

This is in form of (5.1). The MGF is

$$\begin{aligned} M_x(u) &= e^{\log c(e^{\eta+u}) - \log c(e^{\eta})} \\ &= \frac{c(\theta e^u)}{c(\theta)} \end{aligned}$$

Question 5.18

(b) For normal $N(\mu, \sigma^2)$, we have

$$\begin{aligned} p(x|\theta) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2 - \log\sigma\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2} + \frac{ux}{\sigma^2} - (\frac{u^2}{2\sigma^2} + \log\sigma)\right\} \end{aligned}$$

So

$$h(x) = \frac{1}{\sqrt{2\pi}} \quad \eta_1 = -\frac{1}{2\sigma^2} \quad T_1(x) = x^2 \quad \eta_2 = \frac{u}{\sigma^2} \quad T_2(x) = x$$

$$E[g'(x)] = -E[g(x)(-x/\sigma^2 + u/\sigma^2)]$$
$$\sigma^2 E[g'(x)] = -E\left[g(x)(x-u)\right]$$

Let $g(x) = x^2$, then we have

$$\sigma^2 E(2X) = E[X^2(X-\mu)] \Rightarrow 2\sigma^2\mu = E(X^3) - \mu E(X^2)$$
$$\Rightarrow E(X^3) = 2\sigma^2\mu + \mu E(X^2) = 2\sigma^2\mu + \mu(\mu^2 + \sigma^2) = 3\sigma^2\mu + \mu^3$$

Let $g(x) = x^3$, then we have

$$\sigma^2 E(3X^2) = E(X^3(X-\mu)) = E(X^4) - \mu E(X^3)$$

Thus

$$E(X^4) = 3\sigma^2 E(X^2) + \mu E(X^3) = 6\sigma^2\mu^2 + \mu^4 + 3\sigma^4$$

Question 5.28

(a) A is a fixed sample space, so it is not decided by θ. From (5.1),

$$p(x|\theta) = \exp\left(\sum_{i=1}^{s} \eta_i(\theta)T_i(\theta) - B(\theta)\right) h(x)$$

is the unrestricted distribution. Based on this, we have the truncated distribution as

$$\frac{\exp\left(\sum_{i=1}^{s} \eta_i(\theta)T_i(\theta) - B(\theta)\right) I_A(x)h(x)}{p_\theta(A)} = \exp\left(\sum_{i=1}^{s} \eta_i(\theta)T_i(\theta) - B(\theta) - \log p_\theta(A)\right) I_A(x)h(x)$$

with new

$$B'(\theta) = B(\theta) + \log p_\theta(A) \quad h'(\theta) = I_A(x)h(x)$$

Thus the truncated distribution is again in (5.1) form.

(b) We need

$$\{\eta = (\eta_1, \ldots, \eta_s) : \int e^{\sum \eta_i T_i(x)} h(x) d\mu < \infty\}$$

Consider $\exp(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x}$. Truncate it within $x \in [0, 1]$,

$$f^*(x) = \frac{\lambda e^{-\lambda x}}{-e^{-\lambda} + 1} I_{[0,1]}(x)$$

To get

$$\int_0^1 \frac{\lambda e^{-\lambda x}}{-e^{-\lambda} + 1} dx < \infty$$

λ can take any nonzero values, i.e. the natural parameter space is

$$\{\lambda : \lambda \in R,\ \lambda \neq 0\}$$

However for the original distribution, we need $\lambda > 0$ to make $\int_0^\infty \lambda e^{-\lambda x} dx < \infty$, i.e. $\{\lambda : \lambda > 0\}$. The original natural parameter space is a subset of the parameter space of the truncated family.

Question 5.33

(a)

$$f(y) = \frac{\Gamma(1)}{\Gamma(\frac{1}{2}+\frac{\theta}{\pi})\Gamma(\frac{1}{2}-\frac{\theta}{\pi})} y^{-\frac{1}{2}+\frac{\theta}{\pi}}(1-y)^{-\frac{1}{2}-\frac{\theta}{\pi}} = \frac{\cos(\theta)}{\pi} y^{-\frac{1}{2}+\frac{\theta}{\pi}}(1-y)^{-\frac{1}{2}-\frac{\theta}{\pi}}$$

Let $x = \frac{1}{\pi}\log\left(\frac{y}{1-y}\right)$, then $y = \frac{e^{\pi x}}{1+e^{\pi x}}$ and $|J| = \frac{\pi e^{\pi x}}{(1+e^{\pi x})^2}$.

$$\begin{aligned} f(x) &= \frac{\cos(\theta)}{\pi}\left(\frac{e^{\pi x}}{1+e^{\pi x}}\right)^{-\frac{1}{2}+\frac{\theta}{\pi}}\left(1-\frac{e^{\pi x}}{1+e^{\pi x}}\right)^{-\frac{1}{2}-\frac{\theta}{\pi}}\frac{\pi e^{\pi x}}{(1+e^{\pi x})^2} \\ &= \frac{e^{\theta x+\log(\cos\theta)}}{2\cosh(\pi x/2)} \end{aligned}$$

The last equality holds since

$$\cosh(\pi x/2) = \frac{1}{2}\left(e^{\pi x/2} + e^{-\pi x/2}\right)$$

This is in form (5.1).

(b) $s = 1 \quad T = x \quad B(\theta) = -\log(\cos\theta) \quad \eta(\theta) = \theta.$

Apply Theorem 5.8,

$$E_\theta(T(X)) = E(X) = \frac{d}{d\eta}A(\eta) = \tan(\theta) = \mu$$

$$Var_\theta(T(X)) = \frac{d^2}{d\eta^2}A(\eta) = \frac{1}{\cos^2\eta} = 1+\mu^2$$

Another approach to (b) is the following:

$$\begin{aligned} \text{MGF of X} &= E(e^{tX}) \\ &= \int_R h(x)e^{(\theta+t)x-B(\theta)}dx \\ &= e^{-B(\theta)e^{B(\theta+t)}} \\ &= \frac{\cos(\theta)}{\cos(\theta+t)} \qquad \text{for} \quad |\theta+t| < \frac{\pi}{2} \end{aligned}$$

Thus

$$E(X) = \cos(\theta)\left.\frac{\sin(\theta+t)}{\cos^2(\theta+t)}\right|_{t=0} = \tan(\theta)$$

$$
\begin{aligned}
E(X^2) &= \cos\theta \left[\frac{d}{dt} \tan(\theta + t) \sec(\theta + t) \right] \Big|_{t=0} \\
&= \cos\theta [\sec^3(\theta) + (\tan\theta) \sec\theta \tan\theta] \\
&= \sec^2\theta + \tan^2\theta
\end{aligned}
$$

Thus

$$
Var(X) = \sec^2(\theta) = 1 + (EX)^2
$$